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We investigate the relations between the structure of the image and events
in the geometry of the underlying surface. We introduce some elementary
differential geometry and use it to define a coordinate system on the object
based on the lines of survature. Using the coordinate system we can prove
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
ARTIFICIAL INTELLIGENCE LABORATORY

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ZERO-CROSSINGS ON LINES OF CURVATURE

A. Yuille

Abstract. We investigate the relations between the structure of the image and events in the geometry of the undelying surface. We introduce some elementary differential geometry and use it to define a coordinate system on the object based on the lines of curvature. Using this coordinate system we can prove results connecting the extrema, ridges and zero-crossings in the image to geometrical features of the object. We show that extrema of the image typically correspond to points on the surface with zero Gaussian curvature and that parabolic lines often give rise to ridges, or valleys, in the image intensity. We show that directional zero-crossings of the image along the lines of curvature generally correspond to extrema of curvature along such lines. — *want Keywords include — see 14??*

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Zero Crossings on Lines of Curvature

1. Introduction

The aim of this paper is to investigate how much information can be found from an image when the reflectance function is unknown. More precisely we will assume that the image $E(x)$ is determined from the reflectance function $R(p, q)$ and the albedo $a(x, y)$ by the image irradiance equation (Horn 1977)

$$E(x, y) = a(x, y)R(p, q). \qquad (1.1)$$

For a surface $z = f(x, y)$ $p = \partial f/\partial x$ and $q = \partial f/\partial y$. This equation is a good description for the reflectance of many surfaces (Horn 1977). For many surfaces, including most of those considered in shape from shading, the albedo is constant or varies slowly. We will assume it is constant in this paper although many results will not change much if we allow it to vary slowly. Equation (1.1) can be modified to hold for specular surfaces but we will not consider such surfaces in this paper.

For the 1-dimensional case we have

$$E(x) = R(p). \qquad (1.2)$$

Suppose we know nothing about $R(p)$ except that it is continuous. It is still possible to obtain information about the surface by observing that R is a function of p only. If there is a discontinuity in the surface gradient p (i.e. a 2st order discontinuity in the surface) there will be a 1st order discontinuity in the image. In general an Nth order discontinuity in the surface corresponds to a $(N-1)^{th}$ order discontinuity in the image.

We extend this type of arguement to continuous surfaces and prove results relating the distributions of extrema and zero-crossings in the image to features of the underlying surface geometry. To do this we will need to introduce differential geometry concepts such as lines of curvature. We assume that the surfaces of the objects are sufficiently smooth for the lines of curvature to be defined everywhere. We also assume the image is not smoothed by a filter.

In recent years workers in vision have shown considerable interest in the principal lines of curvature of surfaces. For example Curvature Patches have been proposed as a representation for visible surfaces (Brady 1983, Brady and Yuille 1984) and there exist various schemes for dividing objects into parts based on extrema and zeros of curvature

(Brady 1983, Hoffman 1983, Hollerbach 1975). There is also some evidence from line drawings that curves in an image are interpreted as lines of curvature (Stevens 1979) of a perceived surface.

On the other hand it has been suggested that the principal lines of curvature of a surface can only be computed indirectly (for example from edge detection, followed by stereo and surface reconstruction) and with great difficulty. The complexity of the calculations also implies bad numerical behaviour and excessive sensitivity to noise. Recent experimental results (Terzopoulos 1984, Brady and Yuille 1984), admittedly using laser data suggest, however, that these lines can be determined reliably.

This paper is divided into two parts. First we set up a coordinate system based on lines of curvature and relate the image directly to the underlying geometry of the surface using the image irradiance equation (Horn 1977). Then we prove some results concerning zero crossings and the principal lines of curvature. These imply that in certain circumstances we can obtain information about the principal lines of curvature, at least approximately, directly from the image.

In the second part of the paper we consider the relation between the extrema of the image and the parabolic lines of the surface (the lines on which the Gaussian curvature vanishes). Using the line of curvature coordinate system we can generalize results obtained by Koenderink and van Doorn (1980). We show that almost all extrema of the image lie on parabolic lines and that the isophotes (the lines of constant image intensity) point along the lines of curvature when they cut the parabolic lines, independent of the details of the reflectance function. We then argue that parabolic lines typically give rise to ridges, or valleys, in the image intensity. For completeness, in Appendix (1), we include an alternative proof of the first of Koenderink and van Doorn's results which does not rely on the line of curvature coordinate system. We also include an example to show that some extrema of the image intensity do not lie on parabolic lines.

We illustrate these results with the following example. Consider the bottle shown in figure (1). Its curvature will be zero at the head and the base. The neck is initially convex, passes through an inflection point and then becomes concave. Thus the curvature is positive, zero and then negative. The results state that the line with zero curvature will correspond to a ridge, or valley, in the image intensity and that there will be two rings of zero crossings. Furthermore, provided there is not too much foreshortening, these rings are likely to be near the extrema of curvature.

We start by considering various directional zero crossing operators. We show that directional zero crossings do not necessarily correspond to physical, or observable, zero crossings. By physical zero crossings we mean those that correspond to sharp changes in

the image irradiance. We suggest that directional zero crossings are physical only if their direction is along the line of greatest change of the image irradiance and prove a result supporting this choice of operator. This directional operator has been discussed by various authors (Havens and Strikwerda, 1983). More recently Torre and Poggio (Torre and Poggio 1983) have considered it from an alternative point of view. Canny (Canny 1983) has argued for this operator from the standpoint of signal processing.

In section 3 we introduce some mathematical notation and recall some results from Differential Geometry about principal lines of curvature.

Next, in section 4, we introduce the image irradiance equation (Horn 1977). A probabilistic argument shows that the directions of greatest change of the image irradiance are most likely to be along the lines of principal curvature. This suggests that many of the physical zero crossings are directional zero crossings along the principal lines of curvature of the surface that is imaged.

In section 5 we prove some results about the distribution of zero crossings along lines of curvature. Our starting point is the work of Grimson on surface consistency (Grimson 1981). With relatively weak assumptions about the reflectance function (Horn 1977) he was able to produce neccessary and sufficient conditions in one dimension for the occurence of directional zero crossings in the image irradiance in terms of the surface geometry. He then used some probabilistic assumptions about the reflectance surface to extend this result to two dimensions and prove his Surface Consistency Theorem. This theorem was the basis for his theory of surface interpolation.

We present an alternative development of his analysis, and derive some significant extensions. For example, we are able to derive, without probabilistic assumptions, necessary and sufficient conditions for the occurence of directional zero crossings along principal lines of curvature. We call this result *the Line of Curvature Theorem*. It states that *there will be a directional zero crossing along a line of curvature between points where the principal curvature of that line vanishes*. Moreover, depending on the reflectance function, this zero crossing is likely to be near the extremum of the principal curvature.

In conjunction with the results of section 4, the Line of Curvature Theorem suggests that many, if not most, of the physical zero crossings can be associated with points on the lines of principal curvature which are near the extrema of the principal curvatures (see Figure 1). This supports the view that it may be possible to obtain an approximation to the lines of principal curvature directly from the image. It suggests that part boundaries at the extrema of curvature can be found at a low level of visual processing. Finally, it provides additional support for representations of shape based on principal lines of curvature (Brady 1983) and shows that these coordinates are useful for analyzing the image intensities of surfaces.
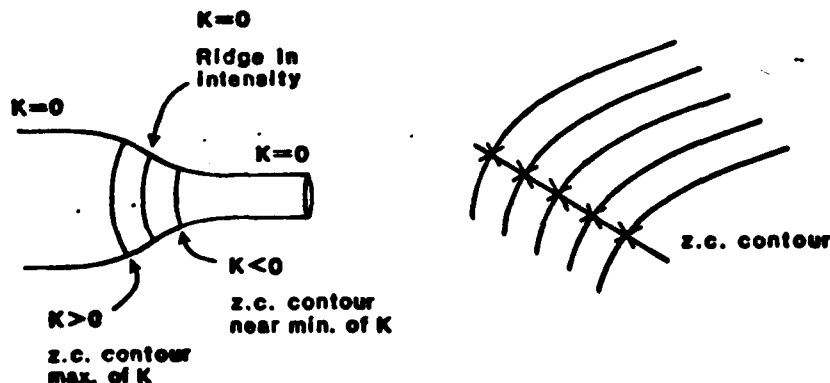
Figure 1. 1. Illustrations of the Line of Curvature Theorem. In (a) we show a bottle. The theorem implies that there is a zero crossing along each of the parallels. Since the bottle is a surface of revolution, it has two rings of zero crossings around the collar. Similarly, (b) is a singly ruled surface for which the zero crossings lie along rulings.

The principle result of the first part of this paper is *the Line of Curvature Theorem*. This derives results about the distribution of directional zero-crossings along the lines of curvature. Assuming that the zero-crossing operator is the directional derivative along the intensity gradient we then argue that these zero-crossings are physical. If another zero-crossing operator is chosen, the Laplacian for example, these arguments are not directly applicable. Instead we argue that the difference in the zero-crossings of different operators (provided they are reasonably similar) lies in the positions of the zero-crossings and not in their existence. If results like the *condition of linear variation* apply (Marr and Hildreth, 1980) the change in the positions of the zero-crossings should be small.

In section 6 we begin the second part of the paper. Using the mathematical machinery already set up it is straightforward to extend the results obtained by Koenderink and van Doorn (1980).The most interesting extension is the claim that parabolic lines often correspond to ridges in the image intensity. In Appendix 1 we provide an alternative proof of some of Koenderink and van Doorn's results which makes no use of the curvature *coordinates*. We also provide an example to show that extrema of the image do not always lie on parabolic lines.

2. Physical Zero Crossings

4

We will now show that not all directional zero crossings can correspond to physical zero crossings. As before we use "physical" to mean those zero crossings that correspond to sharp intensity changes in the image.

Most zero crossing detection operators involve filtering the image with a suitable function, usually a Gaussian. In this section, and for the rest of the paper, we will not consider the effect of the filter in any detail. The purpose of the filter is to smooth the image, thereby isolating irradiance changes at a particular scale. Hence its only effect will be to determine the scale at which the image is analysed and to rule out fluctuations in the image below this scale.

We now show that there are directional zero crossings everywhere in the image. More precisely any circle C_ρ in the image plane with arbitrary centre and radius ρ will contain a directional zero crossing. Note that we are considering continuous images, which have not been digitized.

Let E be the image irradiance. Pick two points in C_ρ with different values of E. At each point there must be a direction in which E does not vary. Join the points by a curve in C_ρ whose tangent vector at each endpoint is in the direction in which E does not vary. There is always an uncountably infinite number of such curves. Now consider the derivative of E along the curve. It is zero at the endpoints and must be non-zero somewhere along the curve since the endpoints have different values of E. Hence it must have an extremum somewhere on the curve and this will be a directional zero crossing in C_ρ. Allowing ρ to tend to zero proves that the set of directional zero crossings is dense in the image plane.

From this argument it is clear that not all directional zero crossings are physically significant. If a curve has unit tangent vector l^μ then a directional zero crossing is a solution of

$$(l^\mu \nabla_\mu)(l^\nu \nabla_\nu)E = 0, \tag{2.1}$$

where ∇_μ denotes the derivative operator. For example in cartesian coordinates $\nabla_\mu = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. We use the summation convention on the suffices μ, ν and so $l^\mu \nabla_\mu$ denotes the scalar product of the vector l^μ with the gradient operator ∇_μ (Hawking and Ellis, 1973). We can write (2.1) as, an alternative version is given in Appendix 2,

$$l^\mu l^\nu (\nabla_\mu \nabla_\nu)E + l^\mu (\nabla_\mu l^\nu) \nabla_\nu E = 0. \tag{2.2}$$

If l^μ is a constant vector, as in Grimson's work, the second term vanishes. However in general it is non zero though it seems plausible to require that the second term in (2.2)
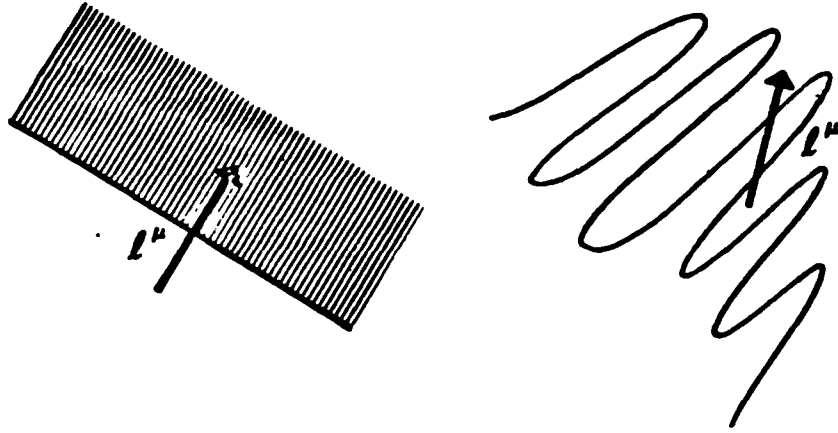
Figure 2. 2. Directional zero crossings. In (a) the unit vector points across an intensity boundary and the zero crossing is physical. In (b) the unit vector points in an arbitrary direction but a curve can be choosen so that it corresponds to a directional zero crossing.

vanishes, or is small. This will ensure that the zero crossing does not depend too much on the derivative of the tangent vector l^μ (in practice this problem rarely arises since we tend to convolve an image in a set of constant directions). To see the importance of this consider figure (2). The physical notion of a directional zero crossing is illustrated in (2.a). The unit vector points from a region of light to a region of dark and is approximately perpendicular to the boundary of these regions. In figure (2.b) however the unit vector l^μ at x points in an arbitrary direction in a region of almost uniform irradiance. Now there are an infinite number of curves though x with tangent l^μ. The result above tells us that it is almost always possible to choose a curve to get a zero crossing at x (given l^μ at x one finds the curve by solving (2.2) for $\nabla_\mu l^\nu$). This zero crossing arises because of the derivative of the tangent vector and is clearly not physical.

We write this constraint as

$$l^\mu(\nabla_\mu l^\nu)\nabla_\nu E = 0. \tag{2.3}$$

The most intuitive zero crossing operator is the directional derivative along the direction of the irradiance gradient (Canny 1983, Havens and Strikwerda 1983, Torre and Poggio 1983). This corresponds to a directional derivative

$$\frac{d}{dt} = \frac{1}{|\nabla_\nu E|}(\nabla_\mu E)\nabla^\mu \tag{2.1}$$

6

where $|\nabla_\nu E|$ is the modulus of the vector $\nabla_\nu E$. In other words we choose l_μ to be $\frac{1}{|\nabla_\nu E|}(\nabla_\mu E)$. Hence

$$\frac{dE}{dt} = |\nabla_\nu E| \tag{2.5}$$

and

$$\frac{d^2 E}{dt^2} = \frac{\partial E}{\partial x_j}\frac{\partial^2 E}{\partial x_j \partial x_k}\frac{\partial E}{\partial x_k}\left(\frac{\partial E}{\partial x_l}\frac{\partial E}{\partial x_l}\right)^{-1}. \tag{2.6}$$

A feature of this operator is that it automatically satisfies our requirement since

$$\left\{\nabla_\mu\left(\frac{1}{|\nabla_\nu E|}(\nabla_\nu E)\right)\right\}\nabla_\nu E = \frac{|\nabla_\nu E|}{2}\nabla_\mu\left\{\frac{1}{|\nabla_\nu E|}\nabla_\nu E \frac{1}{|\nabla_\nu E|}\nabla_\nu E\right\} = 0. \tag{2.7}$$

This result can be used to argue both for $(\nabla_\nu E)\nabla_\nu$ as the directional operator and for our requirement. It means that the zero crossings of this operator will not depend on the derivative of the tangent vector.

Recent work by Canny (Canny 1983) supports this choice of operator. He specifies three performance criteria and looks for the optimal edge detection operator in the presence of background noise. He has also successfully implemented this operator on a large range of images and compared its performance to that of other operators.

3. Lines of Curvature

In this section we briefly recall some of the basic concepts of differential geometry for use in later sections. Consider a surface given by

$$z = f(x, y). \tag{3.1}$$

We can write this as

$$r = (x, y, f(x, y)). \tag{3.2}$$

We will use suffix notation, $(x, y) = (x_1, x_2)$, and write a three dimensional vector l as

$$l = (l_i, l_z) \tag{3.3}$$

splitting it into its components in the (x, y) plane and its component l_z in the z direction. We use the summation convention so if a suffix appears twice it will automatically be summed. For example we use $l_a m_a$ to denote $\sum_a l_a m_a$.

The unit normal to the surface is

$$\mathbf{n} = \left\{1 + \frac{\partial f}{\partial x_j}\frac{\partial f}{\partial x_j}\right\}^{-\frac{1}{2}}\left(\frac{\partial f}{\partial x_i}, -1\right) \tag{3.4}$$

For a surface parametrized by u and v we define the first and second fundamental matrices of the surface by

$$\mathbf{G} = \begin{bmatrix} \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} & \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial u} \\ \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial u} & \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v} \end{bmatrix} \tag{3.5}$$

and

$$\mathbf{D} = \begin{bmatrix} \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial u^2} & \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial u \partial v} \\ \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial u \partial v} & \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial v^2} \end{bmatrix} \tag{3.6}$$

Consider a curve $\mathbf{u} = \mathbf{u}(t)$ on the surface $\mathbf{r} = \mathbf{r}(u, v)$. Let $\mathbf{D}$ and $\mathbf{G}$ be the second and first fundamental matrices of the surface. Then the lines of curvature (Faux and Pratt, 1979) are the solutions of the equation

$$(\mathbf{D} - \kappa\mathbf{G})\frac{d\mathbf{u}}{dt} = 0. \tag{3.7}$$

At any point there are two directions of curvature with curvatures κ_1 and κ_2. They are automatically mutually orthogonal, since they are the eigenvectors of the same matrices, except at umbilic points where $\kappa_1 = \kappa_2$. However at these points it is still possible to choose them to be orthogonal. The Gaussian curvature G is the product of the two curvatures.

$$G = \kappa_1\kappa_2. \tag{3.8}$$

Now choose the parametrization of the surface so that u and v are parameters along the lines of curvature. This corresponds to choosing a "curvature webbing" (Brady 1983) and is illustrated in figure(3). The directions of curvature are $\partial x_i/\partial u$ and $\partial x_i/\partial v$ with curvatures κ_u and κ_v.

It is easy to see from (3.5) and (3.6) that

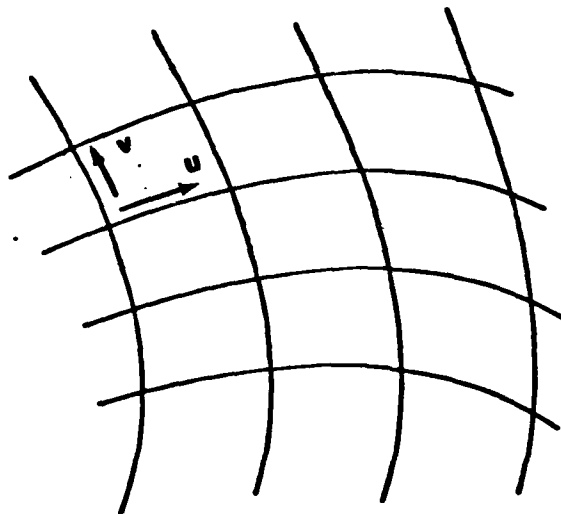$$\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} = 0 \tag{3.9}$$

and

8

Figure 3. 3. A Line of Curvature webbing.

$$n \cdot \frac{\partial^2 r}{\partial u \partial v} = 0. \qquad (3.10)$$

Equations (3.9) and (3.10) demonstrate the chief mathematical advantages of a curvature patch representation (Brady 1983). They show that it diagonalizes the first and second fundamental matrices of the surface, see (3.5) and (3.6), and corresponds to the "flattest" possible local coordinate system.

For a curvature webbing we can write down the principal curvatures directly from (3.7), namely

$$n \cdot \frac{\partial^2 r}{\partial u^2} - \kappa_u \frac{\partial r}{\partial u} \cdot \frac{\partial r}{\partial u} = 0 \qquad (3.11)$$

and

$$n \cdot \frac{\partial^2 r}{\partial v^2} - \kappa_v \frac{\partial r}{\partial v} \cdot \frac{\partial r}{\partial v} = 0. \qquad (3.12)$$

We write $r = (x_i, f)$ and using

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial u} \qquad (3.13)$$

we rewrite (3.9),(3.10),(3.11) and (3.12) as

$$\left\{\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}+\delta_{ij}\right\}\frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial v}=0 \tag{3.14}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial v}=0 \tag{3.15}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial u}+\kappa_u\left(1+\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_k}\right)^{\frac{1}{2}}\left\{\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}+\delta_{ij}\right\}\frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial u}=0 \tag{3.16}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial v}\frac{\partial x_j}{\partial v}+\kappa_v\left(1+\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_k}\right)^{\frac{1}{2}}\left\{\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}+\delta_{ij}\right\}\frac{\partial x_i}{\partial v}\frac{\partial x_j}{\partial v}=0 \tag{3.17}$$

Note that since $\partial x_i/\partial u$ and $\partial x_i/\partial v$ are independent for a curvature webbing, we can use (3.14) and (3.15) to write (3.16) and (3.17) as

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial u}+\kappa_u\left(1+\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_k}\right)^{\frac{1}{2}}\left\{\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}+\delta_{ij}\right\}\frac{\partial x_i}{\partial u}=0 \tag{3.18}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial v}+\kappa_v\left(1+\frac{\partial f}{\partial x_k}\frac{\partial f}{\partial x_k}\right)^{\frac{1}{2}}\left\{\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}+\delta_{ij}\right\}\frac{\partial x_i}{\partial v}=0 \tag{3.19}$$

These are the main results we need about lines of curvature. We finish this section with a few remarks about derivatives on surfaces and Christoffel symbols.

Suppose we have a vector field on a surface defined relative to some coordinate system. Unless the surface is flat, the directions of the axes of the coordinate system will vary over the surface. This also happens if the surface is flat and we choose a coordinate system other than the usual cartesian one, for example Polar Coordinates. In either case we must take the variation of the axes into account when we differentiate. To do this we use Christoffel symbols (Millman and Parker 1977, Hawking and Ellis 1973) denoted by $\Gamma^\mu_{\nu\rho}$. The derivative of a vector l_μ in a coordinate system (x_1, x_2) is then

$$(\nabla_\nu l_\mu)=\frac{\partial l_\mu}{\partial x^\nu}-\Gamma^\mu_{\nu\rho}l^\rho. \tag{3.20}$$

4. The Directions of the Irradiance Gradient

We now argue that the most likely directions of the irradiance gradient are along the lines of curvature.

Consider the image irradiance equation (Horn 1977). We assume orthographic projection onto the (x, y) plane and write the equation as

$$E(x_i) = R(f_i) \qquad (4.1)$$

where $f_i = \partial f / \partial x_i$, $i = 1, 2$. The albedo is assumed constant although the results still hold if it varies slowly.

A vector l_i in the surface has unit length if it satisfies

$$\left\{ \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + \delta_{ij} \right\} l_i l_j = 0 \qquad (4.2)$$

while a unit vector m_i in the image plane obeys

$$m_i m_i = 1. \qquad (4.3)$$

If we consider vectors along the directions of curvature that have unit length *in the surface* we find from (3.15) and (3.16)

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial u} = -\kappa_u \left(1 + \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_k} \right)^{\frac{1}{2}} \qquad (4.4)$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial x_i}{\partial v} \frac{\partial x_j}{\partial v} = -\kappa_v \left(1 + \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_k} \right)^{\frac{1}{2}} \qquad (4.5)$$

We also have

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v} = 0 \qquad (4.6)$$

and the orthogonality condition

$$\left\{ \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + \delta_{ij} \right\} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial v} = 0 \qquad (4.7)$$

Thus $\partial x_i / \partial u$ and $\partial x_i / \partial v$ have unit length in the surface and are the directions in the surface for which the matrix with coefficients $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is extremized (equivalently they are the directions of curvature). Note we can think of these directions as those which extremize

$$\int \left(\left(1 + \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_k} \right)^{\frac{1}{2}} \frac{\partial^2 f}{\partial x_i \partial x_j} l_i l_j \right) d\mathbf{x} + \int \left(\Lambda \left\{ \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + \delta_{ij} \right\} l_i l_j \right) d\mathbf{x} \qquad (4.8)$$

11

where Λ can be thought of as a Lagrange multiplier used to impose orthogonality. The standard equations for lines of curvature can be obtained from (4.8) by using the Euler Lagrange equations.

We can write the derivatives of E in direction l_i as

$$l_i \frac{\partial E}{\partial x_i} = \frac{\partial R}{\partial f_j} \frac{\partial^2 f}{\partial x_i \partial x_j} l_i.$$ (4.9)

The irradiance gradient will be in the direction l_i which maximizes this expression. Now suppose that $\partial R / \partial f_j$ is in a random direction, or that we do not know what the reflectance function is. Then the best choice for l_i is that which extremizes

$$F(l) = \frac{\partial^2 f}{\partial x_i \partial x_j} l_i l_j.$$ (4.10)

where the vectors l_i have unit length in the image plane while the curvature directions $\partial x_i / \partial u$ and $\partial x_i / \partial v$ which extremize $\frac{\partial^2 f}{\partial x_i \partial x_j}$ have unit length in the surface

$$\left\{ \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + \delta_{ij} \right\} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial u} = 1$$ (4.11)

$$\left\{ \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + \delta_{ij} \right\} \frac{\partial x_i}{\partial v} \frac{\partial x_j}{\partial v} = 1.$$ (4.12)

However if we average over the whole surface $\frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial u}$ and $\frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial v}$ should be equal. Hence on average the lengths of $\partial x_i / \partial u$ and $\partial x_i / \partial v$ in the image plane will be equal and they will be the best choices for l_i. Thus the most likely directions for the gradient of the image irradiance is along the lines of curvature. Note that the most likely directions of $\nabla_\nu E$ are not the same as its average direction.

5. The Line of Curvature Theorem

In one dimension the image irradiance equation can be written

$$E(x) = R(p)$$ (5.1)

where $p = df/dx$. Grimson (1981) obtains his theorem in one dimension by differentiating to obtain

$$\frac{dE}{dx} = \frac{dR}{dp} \frac{dp}{dx}$$ (5.2)

and observing that zero-crossings lie between points where $dk/dx = 0$ which typically occur at places where $dp/dx = 0$. In two dimensions this argument fails since the right hand side of (5.2) will now contain several terms. We shall show, however, that it is possible to extend this proof to two dimensions by a special choice of coordinates.

Observe that in the flat two dimensional image space (x_i) and (f_i) are both vectors and equation (4.1) is true whatever coordinate system we choose to represent them in. The coordinate system based on the lines of curvature is a natural choice. We define coordinates $(u_1, u_2) = (u, v)$ and calculate the Christoffel symbols (Millman and Parker 1977, Hawking and Ellis 1973).

$$\Gamma^\mu_{\nu\rho} = \frac{\partial u^\mu}{\partial x_i} \frac{\partial^2 x_i}{\partial u^\nu \partial u^\rho}, \quad \mu, \nu, \rho = 1, 2. \tag{5.3}$$

It should be emphasized that the space we are considering is the flat two dimensional (x, y) plane rather than the surface described by (2.1). This is because zero crossings are found in the image plane.

Denote the differential operator by ∇_μ. In this coordinate system we find (f_i) to be

$$\nabla_\mu f = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right) = (f_u, f_v). \tag{5.4}$$

The second derivatives of f are

$$\nabla_\nu \nabla_\mu f = \frac{\partial}{\partial u^\nu} (\nabla_\mu f) - \Gamma^\rho_{\nu\mu} (\nabla_\rho f). \tag{5.5}$$

Setting $\nu = \mu = 1$ and using (5.3) we obtain

$$\frac{d^2 f}{du^2} = \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial u} \right) - \frac{\partial^2 x_i}{\partial u^2} \frac{\partial f}{\partial x_i}. \tag{5.6}$$

Using (3.13) we can simplify this as

$$\frac{d^2 f}{du^2} = \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial x_i}{\partial u} \frac{\partial x_j}{\partial u}. \tag{5.7}$$

Similarly we find

$$\frac{d^2 f}{dv^2} = \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial x_i}{\partial v} \frac{\partial x_j}{\partial v}, \tag{5.8}$$

and

13

$$\frac{d^2 f}{du\,dv} = \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial v}. \tag{5.9}$$

Now using (3.15) we find

$$\frac{d^2 f}{du\,dv} = 0. \tag{5.10}$$

This result is very important since it enables us to generalize Grimson's one dimensional theorem (Grimson 1981). We write the derivative of E in the u direction as

$$\frac{dE}{du} = \frac{\partial R}{\partial f_u}\frac{df_u}{du} + \frac{\partial R}{\partial f_v}\frac{df_v}{du}. \tag{5.11}$$

Using (5.10) this becomes

$$\frac{dE}{du} = \frac{\partial R}{\partial f_u}\frac{df_u}{du}. \tag{5.12}$$

Then Rolle's Theorem tells us there is a zero of $d^2 E/du^2$, or equivalently a directional zero crossing, in between points where $dE/du = 0$.

We would like a result that is independent of the reflectance map R and, following Grimson, we assume that $\partial R/\partial f_u$ varies slowly compared with df_u/du. This assumption is valid for many forms of the reflectance map (Grimson 1981). Then zeros of dE/du are likely to be zeros of df_u/du. Using (5.7) and (4.4) we see that these occur at

$$\kappa_u = 0. \tag{5.13}$$

We have proved the *Line of Curvature Theorem* which states that *there will be a directional zero crossing along a line of curvature between points where the principal curvature of that line vanishes.* These are the only zero crossings along such lines. Furthermore since the zero crossing occurs at an extremum of dE/du it is likely to be near an extremum of df_u/du. So it is likely to be near an extremum of the curvature κ_u. The more uniform the Reflectance function is the better these assumptions will be. Calculations suggest that these assumptions are most accurate when the gradient of the surface at the extremum of curvature is small. This will be investigated experimentally. Examples of this Theorem are shown in figure (1).

It remains to show that these zero crossings are physical. To do this we can use the results of section (4) to argue that zero crossings along the lines of curvature are likely to

be along the direction of the irradiance gradient. Alternatively we can see whether the lines of curvature satisfy the requirement (2.3). For lines of curvature we calculate

$$l^\mu(\nabla_\mu l^\nu)\nabla_\nu E = \frac{\partial^2 x_i}{\partial u^2}\frac{\partial E}{\partial x_i}.\qquad(5.14)$$

Near extrema of the curvature we expect this to be small since $\partial E/\partial x_i$ is likely to be in the direction $\partial x_i/\partial u$ and the unit length of the tangent vector implies

$$\frac{\partial^2 x_i}{\partial u^2}\frac{\partial x_i}{\partial u} = 0.\qquad(5.15)$$

6. Extrema of the Irradiance

In a recent paper Koenderink and van Doorn (1980) proved that the extrema of illumination of an object lie on the parabolic lines (the lines on which the Gaussian curvature vanishes) and moreover that the isophotes (the lines of constant illumination) cut the parabolic lines at angles which are independent of the position of the source. Using the notation developed in the previous sections we are able to prove stronger results.

We first show that the extrema of illumination almost always lie on the parabolic lines (we demonstrate an exception in Appendix 1) and show how their exact position depends on the reflectance function. Secondly we show that the isophotes lie along the lines of curvature at parabolic lines and hence that angles at which they cut the parabolic lines is a property only of the geometry of the surface and the viewing position and is independent of of the reflectance function and the position of the light source. Finally we argue that these parabolic lines typically correspond to ridges, or valleys, in the image intensity and so it might be possible to compute them directly from the image.

The parabolic lines of a surface are those for which the Gaussian curvature vanishes. Hence they divide the surface into regions of positive and negative Gaussian curvature. Their apparent importance for shape description led the mathematician F. Klein to use them in his attempt to show that the artistic beauty of a face was based on mathematical relations (Hilbert and Cohn-Vossen, 1952).

From (3.8) we see that a necessary and sufficent condition for a line to be parabolic is that at least one of the principal curvatures must vanish on it, for example $\kappa_u = 0$.

The extrema of irradiance occur when

$$\nabla_u E = \nabla_v E = 0.\qquad(6.1)$$

15

From equations (5.12), (5.7) and (3.16), removing terms which are positive definite, we see that (6.1) is equivalent to

$$\frac{\partial R}{\partial f_u}\kappa_u = \frac{\partial R}{\partial f_v}\kappa_v = 0. \tag{6.2}$$

Hence umbilic points with zero Gaussian curvature ($\kappa_u = \kappa_v = 0$), called locally flat points, will give rise to extrema of the irradiance. These points lie on parabolic lines. The most typical extrema are those where $\kappa_u = 0$ and $\frac{\partial R}{\partial f_v} = 0$ (and the same with u and v reversed). These lie on parabolic lines but their exact position will depend on $\frac{\partial R}{\partial f_v}$ and hence on the reflectance function and the positions of the viewer and the source. A final, but unlikely, possibility is when $\frac{\partial R}{\partial f_u} = \frac{\partial R}{\partial f_v} = 0$. This will rarely, if ever, occur on a parabolic line.

Note that extrema of the irradiance do not necessarily correspond to extrema of the surface. These occur at

$$f_i = 0, \tag{6.3}$$

and from (4.1) we see that E will have the same value at all these points

$$E = R(0). \tag{6.4}$$

Whether this corresponds to an extrema of the irradiance or not depends on the explicit reflectance map. For a Lambertian surface

$$R(f_i) = \frac{1 + f_i q_i}{(1 + q_j q_j)^{\frac{1}{2}}(1 + f_k f_k)^{\frac{1}{2}}} \tag{6.5}$$

where the vector q_i corresponds to the direction of the source.

$$R(0) = \frac{1}{(1 + q_i q_i)^{\frac{1}{2}}} \tag{6.6}$$

and

$$R(\delta f_i) = \frac{1 + (\delta f_i)q_i}{(1 + q_j q_j)^{\frac{1}{2}}} + O(\delta^2). \tag{6.7}$$

Near an extremum of f $(\delta f_i)q_i$ can take any sign and so $R(0)$ is not an extremum.

Now suppose the reflectance map depends only on the angle between the surface normal and the observer, perhaps corresponding to a uniform distribution of sources. Then, for some number n,

$$R(f_i) = (1 + f_j f_j)^n \qquad (6.8)$$

which will clearly be extremized at $f_i = 0$.

We now consider the second result of Koenderink and van Doorn (1980). The isophotes are the lines of constant illumination and are similar to the contours on a relief map. At any point their direction is given by the vector l such that

$$l \cdot \nabla E = 0 \qquad (6.9)$$

Suppose we are on a parabolic line given by $\kappa_u = 0$. Using (6.2) and (6.1) it follows that the gradient of the illumination in the direction of the line of curvature parameterized by u is zero, $\nabla_u E = 0$, on the parabolic line. Hence *on parabolic lines the isophotes point along the lines of curvature.* As a corollary of this result we deduce that the angles at which the parabolic lines cut the isophotes is independent of the reflectance function and the position of the source.

We now argue that the parabolic lines generally correspond to ridges or valleys of the image illumination. Again we consider a parabolic line with $\kappa_u = 0$. As shown above this corresponds to $\nabla_u E = 0$. Now consider the values of $\nabla_u E$ on either side of this line. Since the line is parabolic with $\kappa_u = 0$ it follows that κ_u will have different signs on either side of the line, without loss of generality κ_u will be positive on the left side of the line and negative on the right hand side. From (5.12), (5.7) and (3.16) we see that the sign of $\nabla_u E$ on the left and right hand sides of the line will be respectively minus or plus the sign of $\frac{\partial R}{\partial f_u}$. In the generic case $\frac{\partial R}{\partial f_u}$ will be non-zero on the parabolic line and so will have the same sign on either side of the line, for example it could be positive on both sides. Moreover it is likely that this sign will stay the same as we travel along the curve. Provided these assumptions are valid $\nabla_u E$ will be negative on the left side of the line, zero on the line and positive on the right side. Thus, provided the lines of curvature are not too badly behaved, the second derivative of the intensity in the direction of the line of curvature taken at the intersection with the parabolic line will have the same sign along the curve and so the the curve will correspond to a ridge or valley in the image intensity.

7. Conclusion

We have argued that many significant zero crossings lie on the principal lines of curvature near the extrema of curvature. This suggests that it may be a lot easier to

compute some lines of curvature than is commonly believed and that curvature patches are a natural representation for surfaces. Furthermore if the zero crossings pick out the extrema of the principal curvatures it might be possible to make a rough reconstruction of the surface from them.

These results also suggest that part boundaries defined near extrema of the curvature occur near lines of zero crossings. Thus it may be possible to read the part boundaries proposed by Hoffman (1983) and Hoffman and Richards (1984) directly from the shading of an image.

The extensions of Koenderink and van Doorn's results (1980) rederived in section 6 is also interesting as it suggests that the parabolic lines of the surface can be obtained directly from the image. It is suprising that extrema almost always lie on parabolic lines, that these lines are often ridges, or valleys, in the image intensity and that the isophotes point along the directions of curvature when they cut the parabolic lines.

In conclusion it seems that information about the background geometry, and in particular about the lines of curvature and the parabolic lines, can be obtained from the image without too many assumptions about the reflectance function. A coordinate system based on lines of curvature seems a good way to represent a surface.

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Appendix 1.

We first give a short proof showing that the extrema of the image irradiance usually occur on parabolic lines. Then we consider a sphere with a lambertian reflectance function to demonstrate an exception.

We write the image irradiance equation (4.1) in terms of cartesian coordinates

$$E(x, y) = R(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}).$$
(A.1.1)

We differentiate twice by parts, using $p_x = \frac{\partial f}{\partial x}$ and $p_y = \frac{\partial f}{\partial y}$, to obtain

$$\frac{\partial E}{\partial x} = \frac{\partial R}{\partial p_x}\frac{\partial^2 f}{\partial x^2} + \frac{\partial R}{\partial p_y}\frac{\partial^2 f}{\partial x \partial y}$$
(A.1.2)

and

$$\frac{\partial E}{\partial y} = \frac{\partial R}{\partial p_y}\frac{\partial^2 f}{\partial y^2} + \frac{\partial R}{\partial p_x}\frac{\partial^2 f}{\partial x \partial y}.$$
(A.1.3)

In cartesian coordinates the second fundamental form can be written

$$D = \begin{bmatrix} g\frac{\partial^2 f}{\partial x^2} & g\frac{\partial^2 f}{\partial x \partial y} \\ g\frac{\partial^2 f}{\partial x \partial y} & g\frac{\partial^2 f}{\partial y^2} \end{bmatrix} \qquad (A.1.4)$$

where $g = (1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2)^{\frac{1}{2}}$.

An extremum of the image irradiance occurs when

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial y} = 0 \qquad (A.1.5)$$

From equations (A.1.2), (A.1.3) and (A.1.4) it follows that either

$$\frac{\partial R}{\partial p_y} = \frac{\partial R}{\partial p_x} = 0, \qquad (A.1.6)$$

or

$$det\, D = 0. \qquad (A.1.7)$$

If (A.1.7) holds then, by definition, the Gaussian curvature vanishes and so the extremum lies on a parabolic line. This will be the typical situation. However it is also possible that (A.1.6) holds and the extremum lies off the parabolic lines. As an example consider a sphere with a Lambertian reflectance function. The sphere has constant Gaussian curvature and hence no parabolic lines. It must, however, have an extremum of illumination which must therefore correspond to (A.1.7). It is straightforward to check this by an explicit calculation.

Appendix 2.

We illustrate the mathematics of section (2) with the following example.

Consider a coordinate frame (X_1, Y_1) rotated by an angle θ relative to a frame (X, Y). The unit vector in the X_1 direction is $\mathbf{l}$ where

$$\mathbf{l} = (cos\theta, sin\theta) \qquad (A.2.1)$$

We denote the gradient derivative operator by $\underline{\nabla}$ where

$$\underline{\nabla} = \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y} \right). \qquad (A.2.2)$$

Now the transformations between frames is given by

19

$$X_1 = X\cos\theta + Y\sin\theta, \tag{A.2.3}$$

$$Y_1 = Y\cos\theta - X\sin\theta, \tag{A.2.4}$$

and

$$X = X_1\cos\theta - Y_1\sin\theta, \tag{A.2.5}$$

$$Y = Y_1\cos\theta + X_1\sin\theta, \tag{A.2.6}$$

We have

$$\frac{\partial}{\partial X_1} = \underline{l}\cdot\underline{\nabla} = \cos\theta\frac{\partial}{\partial X} + \sin\theta\frac{\partial}{\partial Y}, \tag{A.2.7}$$

Now a directional zero-crossing of E in the X_1 is a solution of

$$\frac{\partial^2 E}{\partial X_1^2} = 0, \tag{A.2.8}$$

or equivalently

$$(\underline{l}\cdot\underline{\nabla})(\underline{l}\cdot\underline{\nabla})E = 0. \tag{A.2.9}$$

We use (A.2.7) to expand this as

$$\frac{\partial^2 E}{\partial X_1^2} = \left(\cos\theta\frac{\partial}{\partial X} + \sin\theta\frac{\partial}{\partial Y}\right)\left(\cos\theta\frac{\partial E}{\partial X} + \sin\theta\frac{\partial E}{\partial Y}\right). \tag{A.2.10}$$

We allow θ to be a function of X and Y, $\theta = \theta(X, Y)$, so the rotation is no longer rigid. Now $\underline{l}(X, Y)$ will be a vector field whose direction depends on its position in space. We can use (A.2.10) to write $\frac{\partial^2 E}{\partial X_1^2}$ as the sum of two terms.

$$\frac{\partial^2 E}{\partial X_1^2} = \begin{aligned}&\left(\cos^2\theta E_{XX} + 2\sin\theta\cos\theta E_{XY} + \sin^2\theta E_{YY}\right) - E_X(\sin\theta\cos\theta\theta_X\\&\quad + \sin^2\theta\theta_Y) + E_Y(\cos^2\theta\theta_X + \sin\theta\cos\theta\theta_Y)\end{aligned} \tag{A.2.11}$$

The second term on the right hand side of (A.2.11) occurs because $\underline{l}$ is a function of (X, Y). In the notation of (2.1) the first term corresponds to $l^\mu l^\nu(\nabla_\mu\nabla_\nu)E$ and the second

to $I^\mu(\nabla_\mu I^\nu)\nabla_\nu E$.In section (2) we argue that unless this term is small a solution of (A.2.8) will be an artifact of the vector field $I(X,Y)$ rather than a physical zero-crossing. We show in equation (2.7) that the second term always vanishes when I is choosen to be along the direction of the gradient of the image intensity.

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